Mathematica Moravica
Vol. **21**, No. **2** (2017), 85–102

Best proximity points for generalized $\alpha-\eta-\psi$-Geraghty proximal contraction mappings

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Abstract. In this paper, we introduce the new notion of generalized $\alpha-\eta-\psi$-Geraghty proximal contraction mappings and prove the existence of the best proximity point for such mappings in $\alpha-\eta$ complete metric spaces. we give an example to illustrate our result. Our result extends some of the results in the literature.

1. Introduction

The purpose of best proximity point theory is to address a problem of finding the distance between two closed sets by using non-self mappings from one set to the other. This problem is known as the proximity point problem. Some mappings on a complete metric space have no fixed point, that is, $d(x, Tx) > 0$ for all $x \in X$. In this case, it is natural to ask the existence and uniqueness of the smallest value of $d(x, Tx)$. This is the main motivation of a best proximity point. This research subject has attracted attention of a number of researchers (see [3, 4, 5, 7, 8, 10, 11]).

Let A and B be two non intersecting subsets of a metric space (X, d). A best proximity point of the mapping T of A into B is a point $u \in A$ satisfying the equality $d(u, Tu) = d(A, B)$, where

$$d(A, B) = inf\{d(x, y) : x \in A, y \in B\}.$$

Let $\mathcal{F}$ be the family of all functions $\beta : [0, \infty) \to [0, 1)$ satisfying the condition:

$$\lim_{n\to\infty} \beta(t_n) = 1 \;\Rightarrow\; \lim_{n\to\infty} t_n = 0.$$

Recently, Chuadchawna et al. introduced a new class of contraction mappings called generalized $\alpha-\eta-\psi$-Geraghty contraction for self mappings. Let Ψ denote the class of all functions $\psi : [0, \infty) \to [0, \infty)$ which satisfy the following conditions:

2000 *Mathematics Subject Classification.* The form is: Primary:47H10; 54H25; ; Secondary: 11J83.

Key words and phrases. Best Proximty Point, α-triangular Proximal admissble with respect to η, generalized $\alpha-\eta-\psi$-Geraghty proximal contraction.

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(a) ψ is nondecreasing;
(b) ψ is continuous;
(c) $\psi(t) = 0 \iff t = 0$.

Definition 1.1. [6] Let (X, d) be a metric space and $\alpha, \eta : X \times X \to [0, \infty)$. A mapping $T : X \to X$ is said to be a generalized $\alpha - \eta - \psi$-Geraghty contraction type mapping if there exists $\beta \in \mathcal{F}$ such that $\alpha(x, y) \geq \eta(x, y)$ implies

$$\psi(d(Tx, Ty)) \leq \beta(\psi(M_T(x, y)))\psi(M_T(x, y)),$$

where

$$M_T(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\} \text{ and } \psi \in \Psi.$$

Definition 1.2. [12] Let $\alpha, \eta : X \times X \to [0, \infty)$ be functions. A mapping $T : X \to X$ is said to be $\alpha-$ orbital admissible with respect to η if for $x \in X$,

$$\alpha(x, Tx) \geq \eta(x, Tx) \implies \alpha(Tx, T^2x) \geq \eta(Tx, T^2x).$$

Definition 1.3. [12] Let $\alpha, \eta : X \times X \to [0, \infty)$ be functions. A mapping $T : X \to X$ is said to be triangular $\alpha-$ orbital admissible with respect to η if

(1) T is $\alpha-$ orbital admissible with respect to η.
(2) $\alpha(x, y) \geq \eta(x, y)$ and $\alpha(y, Ty) \geq \eta(y, Ty)$ imply $\alpha(x, Ty) \geq \eta(x, Ty)$.

Remark 1.1. [6] Every triangular $\alpha-$ admissible mapping is a triangular $\alpha-$ orbital admissible mapping.

Definition 1.4. [9] Let (X, d) be a metric space and $\alpha, \eta : X \times X \to [0, \infty)$. Then X is said to be $\alpha - \eta-$ complete if every Cauchy sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ converges in X.

Example 1.1. [9] Let $X = (0, \infty)$ and $d(x, y) = |x - y|$ be a metric function on X.
Let A be a closed subset of X. Define $\alpha, \eta : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} (x + y)^2, & \text{if } x, y \in A; \\ 0, & \text{otherwise,} \end{cases} \qquad \eta(x, y) = 2xy.$$

Then (X, d) is a $\alpha - \eta$ complete metric space.

Definition 1.5. [9] Let (X, d) be a metric space and $\alpha, \eta : X \times X \to [0, \infty)$. A mapping $T : X \to X$ is said to be $\alpha - \eta-$ continuous mapping if for each $\{x_n\}$ in X with $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ imply $Tx_n \to Tx$ as $n \to \infty$.

Example 1.2. [9] Let $X = [0, \infty)$ and $d(x, y) = |x - y|$ be a metric on X. Assume that $T : X \to X$ and $\alpha, \eta : X \times X \to [0, \infty)$ are defined by:

$$Tx = \begin{cases} x^5, & \text{if } x \in [0\,1]; \\ \sin \pi x + 2, & \text{if } x \in (1, \infty). \end{cases}$$

$$\alpha(x,y) = \begin{cases} x^2 + y^2 + 1, & \text{if } x, y \in [0,1], \\ 0, & \text{otherwise}, \end{cases}$$

$$\eta(x,y) = x^2.$$

Then T is $\alpha - \eta$ continuous map but not a continuous map.

In 2016, Chuadchawna et al. proved the following fixed point theorem for a generalized $\alpha - \eta - \psi -$ Geraghty contraction type mapping.

Theorem 1.1. [6] *Let (X,d) be a metric space. Assume that $\alpha, \eta : X \times X \to [0,\infty)$ be functions and $T : X \to X$ be mapping . Suppose that the following conditions are satisfied:*

i) *(X,d) is an $\alpha - \eta -$complete metric space;*
ii) *T is a generalized $\alpha - \eta - \psi -$ Geraghty contraction type mapping;*
iii) *T is a triangular $\alpha -$ orbital admissible mapping with respect to η;*
iv) *there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$;*
v) *T is an $\alpha - \eta -$ continuous mapping*

Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*.

We refer the reader to [6] for details.

In this paper, we extend the concept of generalized $\alpha - \eta - \psi -$ Geraghty Contraction type mapping to the case of non self mapping. In particular we study the existence of best proximity point for generalized $\alpha - \eta - \psi -$ Geraghty proximal contraction mapping. Several consequences of our obtained results are presented.

2. Preliminaries

Let A and B be two nonempty subsets of a metrics space (X,d). We use the following notations:

$d(A,B) = \inf\{d(a,b) : a \in A, b \in B\}$,
$A_0 = \{a \in A : d(a,b) = d(A,B) \text{ for some } b \in B\}$;
$B_0 = \{b \in A : d(a,b) = d(A,B) \text{ for some } a \in A\}$.

Definition 2.1. An element $x^* \in A$ is said to be a best proximity point of non-self mapping $T : A \to B$ if it satisfies the condition that $d(x^*, Tx^*) = d(A,B)$.

We denote the set of all best proximity points of T by $P_T(A)$,

$$\text{that is}, P_T(A) = \{x \in A : d(x,Tx) = d(A,B)\}.$$

Definition 2.2. [8] Let A and B be two nonempty subsets of a metric space (X,d) and $T : A \to B$ be a mapping. we say that T has $RJ-$ property if for any sequence $\{x_n\} \subset A$,

$$\left.\begin{array}{l} \lim_{n\to\infty} d(x_{n+1}, Tx_n) = d(A,B) \\ \lim_{n\to\infty} x_n = x \end{array}\right\} \implies x \in A_0.$$

We refer the reader to [8] for some more details.

Definition 2.3. [2] A mapping $T : A \to B$ is said to be *proximally increasing* on A if for all $u_1, u_2, x_1, x_2 \in A$,

$$\left.\begin{array}{l} x_1 \preceq x_2 \\ d(u_1, Tx_1) = d(A, B) \\ d(u_2, Tx_2) = d(A, B) \end{array}\right\} \Rightarrow u_1 \preceq u_2,$$

where A and B are nonempty subsets of partially ordered metric space $(X, \preceq, d)$.

Lemma 2.1. [1] *Suppose that (X, d) is a metric space. Let $\{x_n\}$ be a sequence in X such that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence, then there exist an $\epsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $m_k > n_k > k$ such that $d(x_{m_k}, x_{n_k}) \geq \epsilon$, $d(x_{m_k-1}, x_{n_k}) < \epsilon$ and*

i) $\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k+1}) = \epsilon$;
ii) $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon$;
iii) $\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k}) = \epsilon$.

Remark 2.1. By using the hypotheses of Lemma 2.1 and triangular inequality we can show that $\lim_{k\to\infty} d(x_{m_k+1}, x_{n_k+1}) = \epsilon$.

We now introduce the concept of $\alpha-$ orbital proximal admissible with respect to η and triangular $\alpha-$ orbital proximal admissible with respect η in the following definitions.

Definition 2.4. Let $T : A \to B$ be a map and $\alpha, \eta : A \times A \to [0, \infty)$ be functions. we say that T is $\alpha-$ orbital proximal admissible with respect to η if

$$\left.\begin{array}{l} \alpha(x, u) \geq \eta(x, u) \\ d(u, Tx) = d(A, B) \\ d(v, Tu) = d(A, B) \end{array}\right\} \implies \alpha(u, v) \geq \eta(u, v), \quad \text{for all } x, u, v \in A.$$

Definition 2.5. Let $T : A \to B$ be a map and $\alpha, \eta : A \times A \to [0, \infty)$ be functions. we say that T is triangular $\alpha-$ orbital proximal admissible with respect to η if

(1) T is $\alpha-$ orbital proximal admissible with respect to η.

(2) $$\left.\begin{array}{l} \alpha(x, y) \geq \eta(x, y) \\ \alpha(y, u) \geq \eta(y, u) \\ d(u, Ty) = d(A, B) \end{array}\right\} \implies \alpha(x, u) \geq \eta(x, u), \text{ for all } x, y, u \in A.$$

Example 2.1. Let $X = [0, \infty) \times [0, \infty)$ and $d : X \times X \to [0, \infty)$ defined by

$$d((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

Let $A = \{(0, x) : 0 \leq x \leq 1\}$, $B = \{(1, x) : 0 \leq x \leq \frac{1}{3}\}$. Let $\alpha, \eta : A \times A \to [0, \infty)$ defined by

$$\alpha((0, x), (0, y)) = \begin{cases} 2, & \text{if } x, y \leq \frac{1}{3}, \\ 0, & \text{other wise,} \end{cases}$$

$$\eta((0, x), (0, y) = \begin{cases} \frac{1}{3}, & \text{if } x, y \leq \frac{1}{3}, \\ 2, & \text{other wise.} \end{cases}$$

Clearly, $d(A, B) = 1$. We define a mapping $T : A \to B$ by $T(0, x) = (1, \frac{x}{3})$. Then T is triangular $\alpha-$ orbital proximal admissible with respect to η. For, $\alpha((0, x), (0, u)) \geq \eta((0, x), (0, u))$. we have $x, u \leq \frac{1}{3}$. Again let $d((0, v), T(0, u) = d(A, B)$. Then $d((0, v), (1, \frac{u}{3}) = 1$. Which implies $v = \frac{u}{3} \leq \frac{1}{3}$. Thus, we get $u, v \leq \frac{1}{3}$. This implies $\alpha((0, u), (0, v)) \geq \eta((0, u), (0, v))$. Hence T is $\alpha-$ orbital proximal admissible mapping.

Let $\alpha((0, x), (0, y)) \geq \eta((0, x), (0, y)$ and $\alpha((0, y), (0, u)) \geq \eta((0, y), (0, u)$. This implies $x, y, u \leq \frac{1}{3}$. Consequently

$$\alpha((0, x), (0, u)) \geq \eta((0, x), (0, u). \tag{1}$$

By (1) and since T is $\alpha-$ orbital admissible we can conclude that T is triangular $\alpha-$ orbital proximal admissible.

Remark 2.2. Clearly, if $A = B$, T is triangular $\alpha-$ orbital proximal admissible with respect to η implies T is triangular $\alpha-$ orbital admissible with respect to η.

3. Main results

The following proposition is needed to establish the main result.

Proposition 3.1. *Let $T : A \to B$ be a triangular $\alpha-$ oribital proximal admissible mapping. Assume that $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ and $d(x_{n+1}, Tx_n) = d(A, B)$ for all $n \in \mathbb{N}$. Then we have $\alpha(x_n, x_m) \geq \eta(x_n, x_m)$ for all $m, n \in \mathbb{N}$ with $n < m$.*

Proof. Let $m = n + k$. We wish to show for any $k \geq 1$,

$$\alpha(x_n, x_{n+k}) \geq \eta(x_n, x_{n+k}). \tag{2}$$

If $k = 1$, by hypothesis of the proposition, the statement (2) is true. Suppose the statement (2) is true for some $k = t \in \mathbb{N}$. i.e.,

$$\alpha(x_n, x_{n+t}) \geq \eta(x_n, x_{n+t}).$$

Now we want to prove (2) is true for $k = t + 1$, i.e, $\alpha(x_n, x_{n+t+1}) \geq \eta(x_n, x_{n+t+1})$. Now, we have

$$\begin{aligned} \alpha(x_n, x_{n+t}) &\geq \eta(x_n, x_{n+t}); \\ \alpha(x_{n+t}, x_{n+t+1}) &\geq \eta(x_{n+t}, x_{n+t+1}); \end{aligned}$$

$$d(x_{n+t+1}, Tx_{n+t}) \;=\; d(A,B).$$

Since T is $\alpha-$ proximal admissible with respect to η we deduce $\alpha(x_n, x_{n+t+1}) \geq \eta(x_n, x_{n+t+1})$. This implies the statement (2) is true for $k = t+1$. By the principle of Mathematical induction, the statement is true for any $k \geq 1$. Hence $\alpha(x_n, x_m) \geq \eta(x_n, x_m)$ for $n < m$. □

We now introduce the following definition.

Definition 3.1. Let A and B be two nonempty subsets of a metric space (X, d) and $\alpha, \eta : X \times X \to [0, \infty)$ be functions. A mapping $T : A \to B$ is said to be a generalized $\alpha - \eta - \psi$-Geraghty proximal contraction if there exists $\beta \in \mathcal{F}$ such that for all $x, y, u, v \in A$,

$$\left.\begin{array}{l} \alpha(x,y) \geq \eta(x,y) \\ d(u,Tx) = d(A,B) \\ d(v,Ty) = d(A,B) \end{array}\right\} \Longrightarrow$$

$$\psi(d(u,v)) \leq \beta(\psi(M_T(x,y,u,v)))\psi(M_T(x,y,u,v)),$$

where

$$M_T(x,y,u,v) = \max\{d(x,y), d(x,u), d(y,v), \frac{d(x,v)+d(y,u)}{2}\}$$

for any $x, y, u, v \in A$, and $\psi \in \Psi$.

Now we prove the following theorem, which extends, improves and generalizes some results in the literature on best proximity point theorems.

Theorem 3.1. *Let A and B be two nonempty subsets of a metric space (X, d).*
Let $\alpha, \eta : A \times A \to [0, \infty)$ be functions and $T : A \to B$ be a mapping. Suppose the following conditions are satisfied:

i) *(X, d) is an $\alpha - \eta$-complete metric space;*
ii) *T is a generalized $\alpha - \eta - \psi-$Geraghty proximal conntraction type mapping.*
iii) *$T(A_0) \subseteq B_0$ and T is a triangular orbital $\alpha-$ proximal admissible with respect to η.*
iv) *T is $\alpha - \eta$ continuous mapping.*
v) *there exist $x_0, x_1 \in A$ such that $d(x_1, Tx_0) = d(A,B)$ and $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$. Then there exists $x^* \in A_0$ such that $d(x^*, Tx^*) = d(A,B)$.*

Moreover if $\alpha(x,y) \geq \eta(x,y)$ for all $x, y \in P_T(A)$, then x^* is the unique proximity point of T.

Proof. : Let $x_1, x_0 \in A$ be such that $d(x_1, Tx_0) = d(A,B)$ and $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$. Since $x_1 \in A_0$ and $T(A_0) \subseteq B_0$ there exist $x_2 \in A_0$ such that $d(x_2, Tx_1) = d(A,B)$. Now we have

$$\alpha(x_0, y_1) \geq \eta(x_0, x_1);$$
$$d(x_1, Tx_0) = d(A,B);$$

$$d(x_2, Tx_1) = d(A, B).$$

Since T is $\alpha-$ proximal admissible with respect to η, $\alpha(x_2, x_1) \geq \eta(x_2, x_1)$, we have $d(x_2, Tx_1) = d(A, B)$ and $\alpha(x_2, x_1) \geq \eta(x_2, x_1)$.
Continuing this process by induction, we construct a sequence $\{x_n\} \subseteq A_0$ such that

$$\begin{aligned} d(x_{n+1}, Tx_n) &= d(A, B), \\ \alpha(x_n, x_{n+1}) &\geq \eta(x_n, x_{n+1}), \quad \text{for all } n \in \mathbb{N}. \end{aligned} \tag{3}$$

Therefore for any $n \in \mathbb{N}$, we have

$$\begin{aligned} \alpha(x_{n-1}, x_n) &\geq \eta(x_{n-1}, x_n); \\ d(x_n, Tx_{n-1}) &= d(A, B); \\ d(x_{n+1}, Tx_n) &= d(A, B). \end{aligned}$$

Since T is a generalized $\alpha - \eta - \psi$-Geraghty proximal contraction type mapping there exists $\beta \in \mathcal{F}$ such that

$$\begin{aligned} \psi(d(x_n, x_{n+1})) &\leq \beta(\psi(M_T(x_{n-1}, x_n, x_n, x_{n+1})))\psi(M_T(x_{n-1}, x_n, x_n, x_{n+1})); \\ &< \psi(M_T(x_{n-1}, x_n, x_n, x_{n+1})), \end{aligned} \tag{4}$$

where $M_T(x_{n-1}, x_n, x_n, x_{n+1}) = \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1}) + d(x_n, x_n)}{2}\}$, for any $n \in \mathbb{N}$.

From triangular inequality we have

$$d(x_{n-1}, x_{n+1}) \leq d(x_{n-1}, x_n) + d(x_n, x_{n+1}).$$

Thus

$$\frac{d(x_{n-1}, x_{n+1})}{2} \leq \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}.$$

Therefore $M_T(x_{n-1}, x_n, x_n, x_{n+1}) = \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}$, for any $n \in \mathbb{N}$.

If $M_T(x_{n-1}, x_n, x_n, x_{n+1}) = d(x_n, x_{n+1})$, applying (4), we deduce that

$$\begin{aligned} \psi(d(x_n, x_{n+1})) &< \psi(M_T(x_{n-1}, x_n, x_n, x_{n+1}) \\ &= \psi(d(x_n, x_{n+1})), \end{aligned} \tag{5}$$

which is a contradiction. Thus, we conclude that

$$M_T(x_{n-1}, x_n, x_n, x_{n+1}) = d(x_{n-1}, x_n) for all n \in \mathbb{N}. \tag{6}$$

Now from (4) and (6), for all $n \in \mathbb{N}$ we get

$$\psi(d(x_n, x_{n+1})) < \psi(d(x_{n-1}, x_n)).$$

From the nondecreasing property of ψ, for all $n \in \mathbb{N}$ implies that

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n).$$

Hence the sequence $\{d(x_n, x_{n+1})\}$ is nonnegative and nondecreasing. Thus there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. Suppose that there exists $n_0 \in \mathbb{N}$ such that $d(x_{n_0}, x_{n_0+1}) = 0$. This implies that $x_{n_0} = x_{n_0+1}$. Applying (3) we deduce that $d(x_{n_0}, Tx_{n_0}) = d(x_{n_0+1}, Tx_{n_0}) = d(A, B)$. This is the desired result. Now let for any $n \in \mathbb{N}$, $d(x_n, x_{n+1}) \neq 0$. In the sequel, we prove $r = 0$. Contrary let us assume that $r > 0$.

Then from (4) and (6) we have

$$0 \leq \frac{\psi(d(x_n, x_{n+1}))}{\psi(d(x_{n-1}, x_n))} \leq \beta(\psi(d(x_{n-1}, x_n))) < 1.$$

Taking limit as $n \to \infty$ in the above inequality we obtain

$$\lim_{n\to\infty} \beta(\psi(d(x_{n-1}, x_n))) = 1.$$

Since $\beta \in \mathcal{F}$ we get $\lim_{n\to\infty} \psi(d(x_{n-1}, x_n)) = 0$. Again from the properties of ψ, we deduce $\lim_{n\to\infty} d(x_{n-1}, x_n) = 0$. This implies that $r = 0$, which is a contradiction. Therefore $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Now we shall prove that $\{x_n\}$ is a cauchy sequence in (X, d).

Suppose on the contrary $\{x_n\}$ is not Cauchy. Then by Lemma 2.1 , there exist an $\epsilon > 0$ for which we can find sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $m_k > n_k > k$ such that $d(x_{m_k}, x_{n_k}) \geq \epsilon$, $d(x_{m_k-1}, x_{n_k}) < \epsilon$ and the identities (i)-(iii) of Lemma 2.1 and Remark 2.1 are satisfied. Since

$$\begin{aligned} \alpha(x_{n_k}, x_{m_k}) &\geq \eta(x_{n_k}, x_{m_k}); \\ d(x_{n_k+1}, Tx_{n_k}) &= d(A, B); \\ d(x_{m_k+1}, Tx_{m_k}) &= d(A, B). \end{aligned}$$

Since T is $\alpha - \eta - \psi$- Geraghty proximal contraction type mapping, we have

$$\begin{aligned} \psi(d(x_{n_k+1}, x_{m_k+1}) &\leq \beta(\psi(M_T(x_{n_k}, x_{m_k}, x_{n_k+1}, x_{m_k+1})))\cdot \\ &\qquad \cdot \psi(M_T(x_{n_k}, x_{m_k}, x_{n_k+1}, x_{m_k+1})) \\ &< \psi(M_T(x_{n_k}, x_{m_k}, x_{n_k+1}, x_{m_k+1})), \end{aligned} \tag{7}$$

where $M_T(x_{n_k}, x_{m_k}, x_{n_k+1}, x_{m_k+1}) = \max\{d(x_{n_k}, x_{m_k}), d(x_{n_k}, x_{m_k+1}), d(x_{m_k}, x_{m_k+1}), \frac{d(x_{n_k}, x_{m_k+1}) + d(x_{m_k}, x_{n_k+1})}{2}\}$.

Therefore

$$\lim_{k\to\infty} M_T(x_{n_k}, x_{m_k}, x_{n_k+1}, x_{m_k+1}) = \epsilon. \tag{8}$$

By (7) and (8), we have

$$\begin{aligned} 1 &= \frac{\lim_{k\to\infty} \psi(d(x_{n_k+1}, x_{m_k+1}))}{\lim_{k\to\infty} \psi(M_T(x_{n_k}, x_{m_k}, x_{n_k+1}, x_{m_k+1}))} \\ &\leq \lim_{k\to\infty} \beta(\psi(M_T(x_{n_k}, x_{m_k}, x_{n_k+1}, x_{m_k+1}))) \\ &\leq 1, \end{aligned}$$

which implies $\lim_{k\to\infty} \beta(\psi(M_T(x_{n_k}, x_{m_k}, x_{n_k+1}, x_{m_k+1}))) = 1$. Consequently we get $\lim_{k\to\infty} M_T(x_{n_k}, x_{m_k}, x_{n_k+1}, x_{m_k+1}) = 0$. Hence $\epsilon = 0$, which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence in X.

Since (X,d) is $\alpha-\eta$ complete metric space and $\alpha(x_n,x_{n+1})\geq\eta(x_n,x_{n+1})$ for all $n\in\mathbb{N}$, there exists $x^*\in A$ such that $\lim_{n\to\infty}x_n=x^*$. Since T is an $\alpha-\eta-$ continuous, we have $\lim_{n\to\infty}Tx_n=Tx^*$ and $d(A,B)=d(x_{n+1},Tx_n)\to d(x^*,Tx^*)$. Therefore there exists $x^*\in A_0$ such that $d(x^*,Tx*)=d(A,B)$. Hence x^* is best proximity point for the map T.
For uniqueness, let $\alpha(x,y)\geq\eta(x,y)$ for all $x,y\in P_T(A)$.
Suppose that x_1 and x_2 are two best proximity points of T with $x_1\neq x_2$. Therefore

$$\begin{aligned} d(x_1,Tx_1) &= d(A,B);\\ d(x_2,Tx_2) &= d(A,B). \end{aligned}$$

Also, we have

$$\begin{aligned} M_T(x_1,x_2,x_1,x_2) &= \max\Big\{d(x_1,x_2),d(x_1,x_1),d(x_2,x_2),\\ &\qquad \frac{d(x_1,x_2)+d(x_1,x_2)}{2}\Big\}\\ &= d(x_1,x_2). \end{aligned}$$

Since $\alpha(x_1,x_2)\geq\eta(x_1,x_2)$ and T is a generalized $\alpha-\eta-\psi$-Geraghty proximal contraction type mapping, we get

$$\psi(d(x_1,x_2))\leq\beta(\psi(d(x_1,x_2)))\psi(d(x_1,x_2))<\psi(d(x_1,x_2)), \tag{9}$$

which is a contradiction. Hence the best proximity point is unique. □

We provide an example which supports our theorem.

Example 3.1. Let $X=R^2$ and $d:X\times X\to[0,\infty)$ be defined by

$$d((x,y),(x^{'},y^{'})=\sqrt{(x-x^{'})^2+(y-y^{'})^2}.$$

Let

$$\begin{aligned} A &= \{(x,0):0\leq x<\infty\},\\ B &= \{(x,1):0\leq x<\infty\}. \end{aligned}$$

Since $X=R^2$ is a complete metric space it is $\alpha-\eta$-complete metric space and T is also $\alpha-\eta-$ continuous map.

Let $T:A\to B$ be defined by $T(x,0)=(\frac{2x}{x+1},1)$. Let $\alpha,\eta:A\times A\to[0,\infty)$ defined by

$$\alpha((x,0),(y,0))=\begin{cases}3, & \text{if } x,y\in[1,\infty),\\ 1 & \text{other wise.}\end{cases}$$

$$\eta((x,0),(y,0)=\begin{cases}2, & \text{if } x,y\in[1,\infty),\\ 3, & \text{other wise.}\end{cases}$$

Let $\psi:[0,\infty)\to[0,\infty)$ be a function defined by $\psi(t)=\frac{t}{2}$. Then $\psi\in\Psi$

Clearly $d(A,B)=1$, $A=A_0$ and $B=B_0$. Thus $T(A_0)\subseteq B_0$. To show T is Triangular orbital admissible, let $\alpha((x,0),(u,0))\geq\eta((x,0),(u,0))$. This implies $x,u\geq 1$. Moreover, $d((u,0),(\frac{2x}{x+1},1))=1$ and $d((v,0),(\frac{2u}{u+1},1))=1$ imply that $u=\frac{2x}{x+1}$ and $v=\frac{2u}{u+1}$. For $x\geq 1$ we observe $u=\frac{2x}{x+1}\geq 1$ and similarly $v=\frac{2u}{u+1}\geq 1$. Now $u,v\geq 1$ imply that $\alpha((u,0),(v,0))\geq\eta((u,0),(v,0))$. Hence T is α-orbital proximal admissible with respect to η. Furthermore if $\alpha((x,0),(y,0))\geq\eta((x,0),(y,0))$ then $x,y\geq 1$ and $\alpha((0,y),(0,u))\geq\eta((0,y),(0,u))$ imply $u\geq 1$. Consequently $\alpha((x,0),(u,0))\geq\eta((x,0),(u,0))$. Therefore T is triangular $\alpha-$ orbital proximal admissible.

Now we wish to show that T is a generalized $\alpha-\eta-\psi$-Geraghty proximal contraction. i.e, $\exists\beta\in\mathcal{F}$, for each $(x,0),(y,0),(u,0),(v,0)\in A$

$$\left.\begin{array}{l}\alpha((x,0),(y,0))\geq\eta((x,0),(y,0))\\ d((u,0),T(x,0))=d(A,B)\\ d((v,0),T(y,0))=d(A,B)\end{array}\right\}\Longrightarrow$$

$$\psi(d((u,0),(v,0)))\leq\beta(\psi(M_T((x,0),(y,0),(u,0),(v,0))))\cdot$$
$$\cdot\,\psi(M_T((x,0),(y,0),(u,0),(v,0))).$$

Let $\alpha((x,0),(y,0)\geq\eta((x,0),(y,0))$. Then $x,y\in[1,\infty)$. Furthermore $d((u,0),T(x,0))=d(A,B)$ and $d((v,0),T(y,0))=d(A,B)$ imply that $u=\frac{2x}{x+1}$ and $v=\frac{2y}{y+1}$.

Now

$$\begin{aligned}d((u,0),(v,0)) &= d((\frac{2x}{x+1},0),(\frac{2y}{y+1},0))\\ &= |\frac{2x}{x+1}-\frac{2y}{y+1}|\\ &= 2(\frac{|x-y|}{(x+1)(y+1)}).\end{aligned}$$

For $x,y\geq 1$ we can easily observe that $|x-y|+2\leq(x+1)(y+1)$. Thus

$$\frac{d((u,0),(v,0))}{2}\leq\frac{|x-y|}{|x-y|+2}\tag{10}$$

Since $d((x,0),(y,0))=|x-y|\leq M_T((x,0),(y,0),(u,0),(v,0))$ and the map $\gamma(t)=\frac{t}{t+2}$ is non decreasing from (10) we conclude

$$\begin{aligned}\frac{d((u,0),(v,0))}{2} &\leq\frac{M_T((x,0),(y,0),(u,0),(v,0))}{M_T((x,0),(y,0),(u,0),(v,0))+2}\\ &=\frac{1}{\frac{M_T((x,0),(y,0),(u,0),(v,0))}{2}+1}\cdot\\ &\quad\cdot\frac{M_T((x,0),(y,0),(u,0),(v,0))}{2}.\end{aligned}\tag{11}$$

we take $\beta : [0,\infty) \to [0,1)$ defined by

$$\beta(t) = \begin{cases} \frac{1}{t+1}, & \text{if } t \neq 0, \\ 0, & \text{if } t = 0. \end{cases}$$

Thus, from (11) we deduce that there exists $\beta \in \mathcal{F}$ such that

$$\psi(d((u,0),(v,0)) \leq \beta(\psi(M_T((x,0),(y,0),(u,0),(v,0))))\cdot \\ \cdot \psi(M_T((x,0),(y,0),(u,0),(v,0))).$$

Hence T is a generalized $\alpha - \eta - \psi$-Geraghty proximal contraction type mapping.

Since all conditions of Theorem 3.1 are satisfied except uniqueness, T has at least one best proximity point. Note that $x^* = (0,0)$ and $y^* = (1,0)$ are best proximity points of T and we can easily see that $\alpha((0,0),(1,0) < \eta((0,0),(1,0))$.

In the following theorem we replace the continuity of T by some conditions.

Theorem 3.2. *Let A and B be two nonempty and closed subsets of a metric space (X,d).*
Let $\alpha, \eta : A \times A \to [0,\infty)$ be functions and $T : A \to B$ be a mapping. Suppose the following conditions are satisfied:

i) *(X,d) is an $\alpha - \eta$-complete metric space;*
ii) *T is a generalized $\alpha - \eta - \psi-$ Geraghty proximal contraction type mapping.*
iii) *$T(A_0) \subseteq B_0$ and T is a triangular orbital $\alpha-$ proximal admissible with respect to η.*
iv) *T has $RJ-$ property*
v) *If $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ and $x_n \to x \in A$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq \eta(x_{n_k}, x)$ for all $k \in \mathbb{N}$*
vi) *there exist $x_0, x_1 \in A$ such that $d(x_1, Tx_0) = d(A,B)$ and $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$. Then there exists $x^* \in A_0$ such that $d(x^*, Tx^*) = d(A,B)$.*

Moreover if $\alpha(x,y) \geq \eta(x,y)$ for all $x, y \in P_T(A)$, then x^* is the unique proximity point of T.

Proof. Following the proof of Theorem 3.1, there exists a Cauchy sequence $\{x_n\} \subseteq A$ such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ converging to $x^* \in A$. Also $RJ-$ property of T implies that $x^* \in A_0$. Since $T(A_0) \subseteq B_0$, there exists $w \in A_0$ such that $d(w, Tx^*) = d(A,B)$. We need to prove $x^* = w$. On the contrary let us assume that $w \neq x^*$. By (v) there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq \eta(x_{n_k}, x)$ for all $k \in \mathbb{N}$.
For any $k \in \mathbb{N}$, we have $d(x_{n_k+1}, Tx_{n_k}) = d(A,B)$ and $d(w, Tx^*) = d(A,B)$.

Since T is a generalized $\alpha-\eta-\psi-$ Geraghty proximal contraction type mapping, for any $k \in \mathbb{N}$, we have

$$\begin{aligned} \psi(d(x_{n_k+1}, w)) &\leq \beta(\psi(M_T(x_{n_k}, x^*, x_{n_k+1}, w)))\psi(M_T(x_{n_k}, x^*, x_{n_k+1}, w)); \\ &< \psi(M_T(x_{n_k}, x^*, x_{n_k+1}, w)). \end{aligned} \tag{12}$$

Also for any $k \in \mathbb{N}$, we have

$$\begin{aligned} &M_T(x_{n_k}, x^*, x_{n_k+1}, w) \\ &= \max\{d(x_{n_k}, x^*), d(x_{n_k}, x_{n_k+1}), d(x^*, w), \frac{d(x_{n_k}, w) + d(x^*, x_{n_k+1}}{2}\}. \end{aligned}$$

Case I:
Suppose there exist a subsequence $\{x_{n_{k_i}}\} \subset \{x_{n_k}\} \subset \{x_n\}$ such that $M_T(x_{n_{k_i}}, x^*, x_{n_{k_i}+1}, w) = d(x^*, w)$ for all $i \in \mathbb{N}$.

Thus for any $i \in \mathbb{N}$, we have

$$\psi(d(x_{n_{k_i}+1}, w)) \leq \beta(\psi(d(x^*, w)))\psi(d(x^*, w)). \tag{13}$$

Taking limit in (13) as $i \to \infty$ implies that $\beta(\psi(d(x^*, w))) = 1$. Which implies that $d(x^*, w) = 0$, which is a contradiction.

Case II:
Suppose there exist a subsequence $\{x_{n_{k_i}}\} \subset \{x_{n_k}\} \subset \{x_n\}$ such that

$$M_T(x_{n_{k_i}}, x^*, x_{n_{k_i}+1}, w) = \frac{d(x_{n_{k_i}}, w) + d(x^*, x_{n_{k_i}+1})}{2} \text{ for all } i \in \mathbb{N}.$$

Letting $i \to \infty$ we get

$$\lim_{i\to\infty} M_T(x_{n_{k_i}}, x^*, x_{n_{k_i}+1}, w) = \frac{d(x^*, w)}{2}.$$

Thus, we have

$$\psi(d(x^*, w)) \leq \beta(\psi(\frac{d(x^*, w)}{2}))) \tag{14}$$

$$< \psi(\frac{d(x^*, w)}{2}). \tag{15}$$

Since ψ is non-decreasing, it follows that, $d(x^*, w) < \frac{d(x^*,w)}{2}$. This is a contradiction.

Case III:
Suppose that there exists $t \in \mathbb{N}$ such that

$$M_T(x_{n_{k_i}}, x^*, x_{n_{k_i}+1}, w) = \max\{d(x_{n_{k_i}}, x^*), d(x_{n_{k_i}}, x_{n_{k_i}+1})\} \text{ for all } i \geq t. \tag{16}$$

From (7) and above result, and by taking the limit as $i \to \infty$, we deduce that

$d(x^*, w) = 0$. This is a contradiction. Therefore $x^* = w$, which implies that

$$d(x^*, Tx^*) = d(w, Tx^*) = d(A, B).$$

Hence x^* is the best proximity point of T. □

If in Theorem 3.1 or Theorem 3.2 we take $\eta(x, y) = 1$ and $\psi(t) = t$, then we have the following corollary.

Corollary 3.1. *Let A and B be two nonempty subsets of a metric space (X, d). Let $\alpha : A \times A \to [0, \infty)$ be function and $T : A \to B$ be a mapping. Suppose the following conditions are satisfied:*

i) *T is a generalized $\alpha-$ Geraghty proximal contraction type mapping, that is*

$$\left.\begin{array}{l} \alpha(x,y) \geq 1 \\ d(u,Tx) = d(A,B) \\ d(v,Ty) = d(A,B) \end{array}\right\} \Longrightarrow d(u,v) \leq \beta(M_T(x,y,u,v))M_T(x,y,u,v),$$

where

$$M_T(x,y,u,v) = \max\{d(x,y), d(x,u), d(y,v), \frac{d(x,v)+d(y,u)}{2}\},$$

for any $x, y, u, v \in A$.

ii) *The conditions (i), (iii)-(v) of Theorem 3.1 or 3.2 are satisfied.*

Then there exists $x^* \in A_0$ such that $d(x^*, Tx^*) = d(A, B)$. Moreover if $\alpha(x, y) \geq 1$ for all $x, y \in P_T(A)$, then x^* is the unique proximity point of T.

4. Consequences

We start this section with the following definition.

Definition 4.1. Let A and B be two nonempty subsets of a metric space (X, d) and $\alpha, \eta : X \times X \to [0, \infty)$ be functions. A mapping $T : A \to B$ is said to be a $\alpha - \eta - \psi$-Geraghty proximal contraction if there exists $\beta \in \mathcal{F}$ such that for all $x, y, u, v \in A$,

$$\left.\begin{array}{l} \alpha(x,y) \geq \eta(x,y) \\ d(u,Tx) = d(A,B) \\ d(v,Ty) = d(A,B) \end{array}\right\} \Longrightarrow \psi(d(u,v)) \leq \beta(\psi(d(x,y)))\psi(d(x,y)),$$

where $\psi \in \Psi$.

Theorem 4.1. *Let A and B be two nonempty subsets of a metric space (X, d). Let $\alpha, \eta : A \times A \to [0, \infty)$ be functions and $T : A \to B$ be a mapping. Suppose the following conditions are satisfied:*

i) *(X, d) is an $\alpha - \eta$-complete metric space;*

ii) *T is a generalized $\alpha - \eta - \psi-$ Geraghty proximal contraction type mapping.*

iii) *$T(A_0) \subseteq B_0$ and T is a triangular orbital $\alpha-$ proximal admissible with respect to η.*
iv) *T is $\alpha - \eta$ continuous mapping.*
v) *there exist $x_0, x_1 \in A$ such that $d(x_1, Tx_0) = d(A, B)$ and $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$. Then there exists $x^* \in A_0$ such that $d(x^*, Tx^*) = d(A, B)$.*

Moreover if $\alpha(x, y) \geq \eta(x, y)$ for all $x, y \in P_T(A)$, then x^* is the unique proximity point of T.

Proof. Let $x_0, x_1 \in A$ such that $d(x_1, Tx_0) = d(A, B)$ and $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$. As in the proof of Theorem 3.1 we construct a sequence $\{x_n\}$ in A_0 such that

(17) $d(x_{n+1}, Tx_n) = d(A, B)$ and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$,

and converging to some $x^* \in A_0$. Since T is $\alpha - \eta-$ continuous mapping, we have

$$d(A, B) = d(x_{n+1}, Tx_n) \to d(x^*, Tx^*) \text{ as } n \to \infty.$$

Hence T has best proximity point.

Uniqueness of this best proximity point is proved as in Theorem 3.1. □

Theorem 4.2. *Let A and B be two nonempty subsets of a metric space (X, d).*
Let $\alpha, \eta : A \times A \to [0, \infty)$ be functions and $T : A \to B$ be a mapping. Suppose the following conditions are satisfied:

i) *(X, d) is an $\alpha - \eta$-complete metric space;*
ii) *T is an $\alpha - \eta - \psi-$ Geraghty proximal conntraction type mapping.*
iii) *$T(A_0) \subseteq B_0$ and T is a triangular orbital $\alpha-$ proximal admissible with respect to η.*
iv) *T has $RJ-$ property*
v) *If $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ and $x_n \to x \in A$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq \eta(x_{n_k}, x)$ for all $k \in \mathbb{N}$*
vi) *there exist $x_0, x_1 \in A$ such that $d(x_1, Tx_0) = d(A, B)$ and $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$. Then there exists $x^* \in A_0$ such that $d(x^*, Tx^*) = d(A, B)$*

Moreover if $\alpha(x, y) \geq \eta(x, y)$ for all $x, y \in P_T(A)$, then x^* is the unique proximity point of T.

Proof. Let $x_0, x_1 \in A$ such that $d(x_1, Tx_0) = d(A, B)$ and $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$. As in the proof of Theorem 3.1 we construct a sequence $\{x_n\}$ in A_0 such that

(18) $d(x_{n+1}, Tx_n) = d(A, B)$ and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$,

and converging to some $x^* \in A_0$. By(v) there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq \eta(x_{n_k}, x)$ for all $k \in \mathbb{N}$. Further more $RJ-$ property of T implies that $x^* \in A_0$. Since $T(A_0) \subseteq B_0$, there exists $w \in A_0$ such that $d(w, Tx^*) = d(A, B)$. We need to prove $x^* = w$. On the contrary let us assume that $w \neq x^*$. For any $k \in \mathbb{N}$, we have $d(x_{n_k+1}, Tx_{n_k}) = d(A, B)$. Now for all $k \in \mathbb{N}$ we have

$$\left.\begin{array}{r}\alpha(x_{n_k}, x^*) \geq \eta(x_{n_k}, x^*) \\ d(x_{n_k+1}, Tx_{n_k}) = d(A, B) \\ d(w, Tx^*) = d(A, B)\end{array}\right\}$$

Since T is $\alpha - \eta - \psi$-Geraghty proximal contraction there exists $\beta \in \mathcal{F}$ such that

$$\begin{aligned}\psi(d(x_{n_k+1}, w)) &\leq \beta(\psi(d(x_{n_k,x^*}))) \\ &< \psi(d(x_{n_k}, x^*))\end{aligned} \tag{19}$$

Letting $k \to \infty$ in (19) we get $\psi(d(x^*, w)) \leq 0$. Thus $\psi(d(x^*, w)) = 0$. This implies $d(x^*, w) = 0$. This is a contradiction. Hence $x^* = w$. Uniqueness of x^* is proved as in the Theorem 3.1. □

Corollary 4.1. *Let $(X, \preceq)$ be a partial ordered set and suppose there exists a metric d such that $(X, \preceq, d)$ complete. Let A, B be two nonempty closed subsets of X. Suppose $T : A \to B$ be a mapping. Assume that the following conditions are satisfied:*

i) *there exists $\beta \in \mathcal{F}$ such that for all $x, y, u, v \in A$,*

$$\left.\begin{array}{l}x \preceq y \\ d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B)\end{array}\right\} \implies \psi(d(u, v)) \leq \beta(\psi(d(x, y)))\psi(d(x, y)),$$

where $\psi \in \Psi$.

ii) *there exist $x_0, x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A, B)$ and $x_0 \preceq x_1$.*

iii) *T is proximal nondecreasing and $T(A_0) \subseteq B_0$.*

iv) *either T is continuous or T has $RJ-$ property and if $\{x_n\}$ is a non decreasing sequence with $x_n \to x$ as $n \to \infty$, there exists a sub sequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \preceq x$ for all $k \in \mathbb{N}$.*

Then there exists $x^* \in A_0$ such that $d(x_0, Tx_0) = d(A, B)$.
Moreover, if x and y are comparable for all $x, y \in P_T(A)$, then x^* is the unique proximity point of T.

Proof. Define functions $\alpha, \eta : A \times A \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 2, & \text{if } x \preceq y, \\ \frac{3}{4}, & \text{otherwise.} \end{cases}$$

$$\eta(x, y) = \begin{cases} 1, & \text{if } x \preceq y, \\ 2, & \text{otherwise.} \end{cases}$$

Let $x, y, u, v \in A$ with $\alpha(x, y) \geq \eta(x, y)$, $d(u, Tx) = d(A, B)$ and $d(v, Ty) = d(A, B)$. This implies $x \preceq y$. By (i) $\psi(d(u, v)) \leq \beta(\psi(d(x, y)))\psi(d(x, y))$. This implies that T is an $\alpha - \eta - \psi-$ Geraghty proximal contraction.

Since X is complete space X is $\alpha-\eta-$ complete space. By (ii) and definition of α, η there exist $x_0, x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A, B)$ and $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$.

Let $\alpha(x, u) \geq \eta(x, u)$, $d(u, Tx) = d(A, B)$ and $d(v, Tu) = d(A, B)$. This implies $x \preceq u$. Since T is proximal nondecreasing we get that $u \preceq v$. Then $\alpha(u, v) \geq \eta(u, v)$. Furthermore, let $\alpha(x, y) \geq \eta(x, y), \alpha(y, u) \geq \eta(y, u)$ and $d(u, Ty) = d(A, B)$. This implies that $x \preceq y$ and $y \preceq u$. consequently $x \preceq u$. Thus $\alpha(x, u) \geq \eta(x, u)$. Therefore T is triangular $\alpha-$orbital proximal admissible. Thus all conditions of either Theorem 4.1 or Theorem 4.2 satisfied. Hence T has best proximity point.
Moreover x and y are comparable for all $x, y \in P_T(A)$ imply that either $\alpha(x, y) \geq \eta(x, y)$ or $\alpha(y, x) \geq \eta(y, x)$. Thus similar to Theorem 3.1 we get that x^* is unique. □

5. APPLICATION IN FIXED POINT THEORY

As an application of our results, we prove this fixed point theorem which is proved by Chuadchawna et al. [6] as follows.

Theorem 5.1. *Let (X, d) be a metric space. Assume that $\alpha, \eta : X \times X \to [0, \infty)$ be functions and $T : X \to X$ be mapping . Suppose that the following conditions are satisfied:*

i) *(X, d) is an $\alpha-\eta-$complete metric space;*
ii) *T is a generalized $\alpha-\eta-\psi-$ Geraghty contraction type mapping;*
iii) *T is a triangular $\alpha-$ orbital admissible mapping with respect to η;*
iv) *there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$;*
v) *T is an $\alpha-\eta-$ continuous mapping*

Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*.

Proof. Let $A = B = X$ in Theorem 3.1. First we prove that T is a generalized $\alpha-\eta-\psi$-Geraghty proximal contraction type map. Let $x, y, u, v \in X$, satisfying the following conditions

$$\begin{cases} \alpha(x, y) \geq \eta(x, y), \\ d(u, Tx) = d(A, B), \\ d(v, Ty) = d(A, B). \end{cases}$$

Since $d(A, B) = 0$, we have $u = Tx$ and $v = Ty$. Since T is generalized $\alpha-\eta-\psi$-Geraghty Contraction mapping, which implies that

$$\psi(d(u, v)) = \psi(d(Tx, Ty)) \leq \beta(\psi(M_T(x, y)))\psi(M_T(x, y))$$

where

$$\begin{aligned} M_T(x, y) &= \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\} \\ &= \max\{d(x, y), d(x, u), d(y, v), \frac{d(x, v) + d(y, u)}{2}\} \end{aligned}$$

$$= \quad M_T(x,y,u,v).$$

Therefore

$$\psi(d(u,v)) \leq \beta(\psi(M_T(x,y,u,v)))\psi(M_T(x,y,u,v)),$$

which implies that T is a generalized $\alpha{-}\eta{-}\psi$-Geraghty proximal contraction type map.

$$\text{Let} \begin{cases} \alpha(x,u) \geq \eta(x,u), \\ d(u,Tx) = d(A,B), \\ d(v,Tu) = d(A,B). \end{cases}$$

Since $d(A,B) = 0$, we have $u = Tx, v = Tu = T^2x$. Thus $\alpha(x,Tx) \geq \eta(x,Tx)$. Orbital admissible property of T implies that

$$\alpha(u,v) = \alpha(Tx,T^2x) \geq \eta(Tx,T^2x) = \eta(u,v).$$

Therefore T is α-orbtal proximal admissible with respect to η. Moreover, let

$$\begin{cases} \alpha(x,y) \geq \eta(x,y), \\ \alpha(y,u) \geq \eta(y,u), \\ d(u,Ty) = d(A,B) = 0. \end{cases}$$

This implies $u = Ty$. T is a triangular $\alpha-$orbital admissible property implies that $\alpha(x,u) = \alpha(x,Ty) \geq \eta(x,Ty) = \eta(x,u)$. Therefore T is a triangular $\alpha-$ orbital admissible with respect to η. Applying Condition (iv) there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq \eta(x_0,Tx_0)$. Let $x_1 = Tx_0$, thus $\alpha(x_0,x_1) \geq \eta(x_0,Tx_0)$ and $d(x_1,Tx_0) = d(Tx_0,Tx_0) = d(A,B) = 0$.

All conditions of Theorem 3.1 are satisfied. Consequently there exists $x^* \in X$ such that $d(x^*,Tx^*) = 0$. This implies $x^* = Tx^*$. □

If $\eta(x,y) = 1$ for all $x,y \in A = X$, and in view of Remark 1.1, we get the following corollary proved by Karapinar [10].

Corollary 5.1. *Let (X,d) be a complete metric space. Assume that $\alpha : X \times X \to [0,\infty)$ be functions and $T : X \to X$ be mapping . Suppose that the following conditions are satisfied:*

i) *T is a generalized $\alpha - \psi-$ Geraghty contraction type mapping;*
ii) *T is a triangular $\alpha-$ admissible mapping;*
iii) *there exists $x_1 \in X$ such that $\alpha(x_1,Tx_1) \geq 1$;*
iv) *T is continuous mapping.*

Then T has a fixed point $x^* \in X$ and $\{T^nx_1\}$ converges to x^*.

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